Modelling and computations in elastoplasticity

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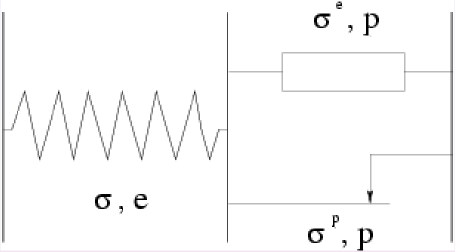
Outline

Time dependent 2D problem in Matlab

Figure: elastoplastic zones: green - elastic, pink - plastic

Kinematic hardening model



$$\varepsilon = \mathrm{e} + p$$
$$\sigma = \sigma^e + \sigma^p$$
$$\sigma^e = \mathcal{H} p$$
$$\sigma = \mathbb{C}\,\mathrm{e}$$
$$\sigma^p \in Z$$
$$\langle \dot{p}, q - \sigma^p \rangle \le 0 \quad \text{for all } q \in Z.$$

Hysteresis property of the kinematic hardening model

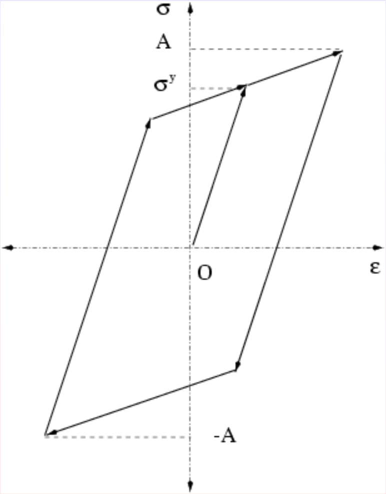


Figure: Stress-strain relation in case of linear kinematic hardening model and the cyclic stress $\sigma = A\sin(t)$.

Motivation for the multi-yield model

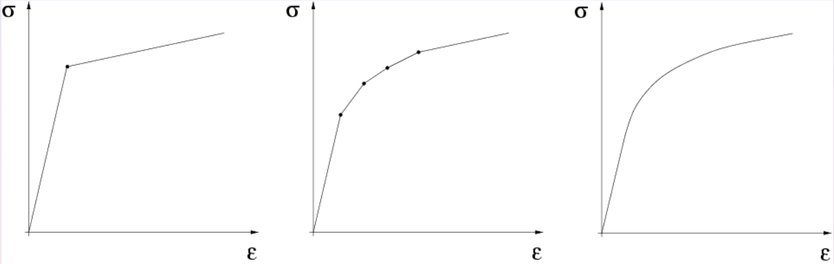
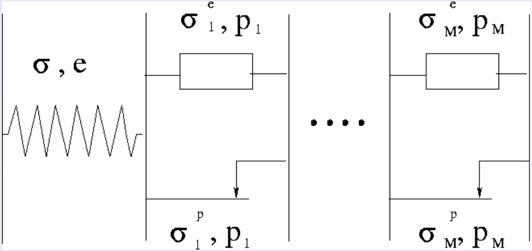


Figure: single-yield (left), multi-yield (middle) and realistic model (right) - stress-strain relation.

The M-yield hardening model



$$
\begin{aligned}
&\varepsilon = \mathrm{e} + p, \quad p = \sum_{r=1}^{M} p_r, \\
&\sigma = \sigma_r^e + \sigma_r^p \quad \text{for all } r = 1, \dots, M, \\
&\sigma_r^p \in Z_r, \\
&\langle \dot{p}_r, q_r - \sigma_r^p \rangle \leq 0 \quad \text{for all } q_r \in Z_r, r = 1, \dots, M, \\
&\sigma = \mathbb{C}\,\mathrm{e}, \\
&\sigma_r^e = \mathcal{H}_r p_r, \quad r = 1, \dots, M.
\end{aligned}
$$

Hysteresis property of the 2-yield hardening model

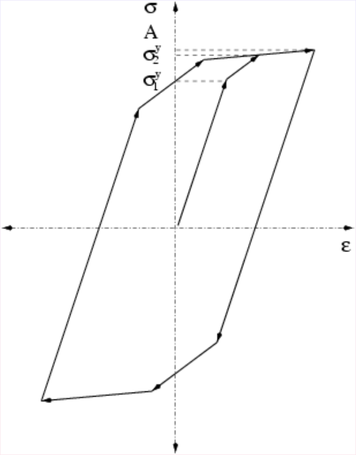


Figure: Stress-strain relation in case of two-yield model and cyclic stress $\sigma = A \sin(t)$.

Books on hysteresis

- Visintin, A., Differential models of hysteresis, Springer, 1994
- Brokate, M. and Sprekels, J., Hysteresis and Phase Transitions, Springer-Verlag New York, 1996
- Krejčí, P., Hysteresis, Convexity and Dissipation in Hyperbolic Equations, GAKUTO International Series, Mathematical Sciences and Applications, 1996

Outline

Variational inequality

We collect vectors of functions

$$w = (u, (p_r)_{r\in I}), \quad z = (v, (q_r)_{r\in I}).$$

to obtain

Problem (BVP of quasi-static multi-surface elastoplasticity)

For given $\ell \in H^1(0, T; \mathcal{H}^*)$ *with* $\ell(0) = 0$,
find $w \in H^1(0, T; \mathcal{H})$ *with* $w(0) = 0$, *such that*

$$a(w(t), z - \dot{w}(t)) + \psi(z) - \psi(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle, \quad \text{for all } z \in \mathcal{H},$$

holds for almost all $t \in (0, T)$.

Variational inequality

A bilinear form $a(\cdot,\cdot)$, a linear functional $\ell(\cdot)$ and a nonlinear functional $\psi(\cdot)$ are defined as

$$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \quad a(w,z) = \int_\Omega \mathbb{C}(\varepsilon(u) - \sum_{r\in I} p_r) : (\varepsilon(v) - \sum_{r\in I} q_r) \, \mathrm{d}x + $$

$$+ \sum_{r\in I} \int_\Omega \mathcal{H}_r p_r : q_r \, \mathrm{d}x,$$

$$\ell(t) : \mathcal{H} \to \mathbb{R}, \quad \langle \ell(t), z \rangle = \int_\Omega f(t) \cdot v \, \mathrm{d}x + \int_{\Gamma_N} g(t) \cdot v \, \mathrm{d}s,$$

$$\psi : \mathcal{H} \to \mathbb{R}, \quad \psi(z) = \sum_{r\in I} \int_\Omega \mathcal{D}_r(q_r) \, \mathrm{d}x.$$

and $\mathcal{H} = H^1_D(\Omega) \times \prod_{r\in I} Q$.

Further reading

- Glowinskii, R., Lions J. L. and Trémolières R., Numerical analysis of Variational Inequalities, North-Holland, Amsterdam, 1981
- Han, W. and Reddy, B., Plasticity: Mathematical Theory and Numerical Analysis, Springer-Verlag New York, 1999

Material assumptions

We pose the natural assumption that the elastic and hardening tensors are symmetric and positive definite,

$$\begin{aligned} \xi : \mathbb{C}\lambda = \mathbb{C}\xi : \lambda \quad & \text{for all } \xi, \lambda \in \mathbb{R}^{d\times d}, \\ \xi : \mathcal{H}_r\lambda = \mathcal{H}_r\xi : \lambda \quad & \text{for all } \xi, \lambda \in \mathbb{R}^{d\times d}, r = 1, \dots, M, \end{aligned} \tag{1}$$

and there exist constants $c, h_r > 0$ such that

$$\begin{aligned} \mathbb{C}\xi : \xi \geq c||\xi||^2 \quad & \text{for all } \xi \in \mathbb{R}^{d\times d}, \\ \mathcal{H}_r\xi : \xi \geq h_r||\xi||^2 \quad & \text{for all } \xi \in \mathbb{R}^{d\times d}, r = 1, \dots, M. \end{aligned} \tag{2}$$

Abstract theorem on solvability

Theorem

*Assume that (**??**) and (**??**) hold, let $\ell \in H^1(0, T; \mathcal{H}^*)$ with $\ell(0) = 0$. Then there exists a unique solution $w \in H^1(0, T; \mathcal{H})$ of BVP of quasi-static multi-surface elastoplasticity.*

based on

Abstract theorem on solvability

Theorem (Han, Reddy, 1999)

Let $\mathcal{H}$ be a Hilbert space, $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form that is symmetric, bounded, and $\mathcal{H}$-elliptic; $\ell \in H^1(0, T; \mathcal{H}^*)$ with $\ell(0) = 0$; and $\psi : \mathcal{H} \to \mathbb{R}$ nonnegative, convex, positively homogeneous, and Lipschitz continuous. Then there exists a unique $w \in H^1(0, T; \mathcal{H})$ with $w(0) = 0$ which satisfies the variational inequality

$$a(w(t), z - \dot{w}(t)) + \psi(z) - \psi(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle , \quad \text{for all } z \in \mathcal{H},$$

for almost all $t \in (0, T)$.

Remark on ellipticity

To prove that

$$a(w,z) = \int_\Omega \mathbb{C}(\varepsilon(u) - \sum_{r\in I} p_r) : (\varepsilon(v) - \sum_{r\in I} q_r)\,\mathrm{d}x + \sum_{r\in I} \int_\Omega \mathcal{H}_r p_r : q_r\,\mathrm{d}x,$$

is elliptic, the following partial result is important:

Problem

To determine the largest constant $k(M)$, $M \in \mathcal{N}$, such that

$$\Big(x_0 - \sum_{r=1}^{M} x_r\Big)^2 + \sum_{r=1}^{M} x_r^2 \geq k(M) \sum_{r=0}^{M} x_r^2 \tag{3}$$

holds for all $x_0, x_1, \ldots, x_M \in \mathbb{R}$.

Algebraic inequality

We refolmulate

$$\left(x_0 - \sum_{r=1}^{M} x_r\right)^2 + \sum_{r=1}^{M} x_r^2 = x^T A x, \tag{4}$$

where

$$A = D + a \otimes a\,, \quad D = \mathrm{diag}(0, 1, \ldots, 1)\,, \quad a = (1, -1, \ldots, -1)\,. \tag{5}$$

Thus, the optimal constant $k(M)$ is equal to the smallest eigenvalue of A!

Algebraic inequality

The analytical computation shows

$$k(M) = \lambda_{min} = 1 + \frac{M}{2} - \frac{1}{2}\sqrt{4M + M^2}$$

Properties:

$$\lim_{M \to \infty} k(M) = 0$$

and

$$\lim_{M \to \infty} Mk(M) = 1$$

Outline

Backward Euler scheme

In the first time step t_1, the time derivative $\dot{x}(t_1)$ is approximated by the backward Euler method as

$$\dot{X}^1 = \frac{X^1 - X^0}{k_1},$$

where $X^0 = 0$. The Hilbert space $\mathcal{H}$ is approximated by the conforming finite element (FEM) subspace

$$\mathcal{S} = \mathcal{S}^1_D(\mathcal{T}) \times \prod_{r \in I} \operatorname{dev}(\mathcal{S}^0(\mathcal{T})^{d\times d}_{\mathrm{sym}}),$$

which is a product space of $\mathcal{T}$- piecewise affine functions that are zero on Γ_D by

$$\mathcal{S}^1_D(\mathcal{T}) := \{v \in H^1_D(\Omega) : \forall T \in \mathcal{T}, v|_T \in \mathcal{P}_1(T)^d\}.$$

($\mathcal{P}_1(T)$ denotes the affine functions on T) and the space of $\mathcal{T}$- piecewise constant functions

$$\operatorname{dev}(\mathcal{S}^0(\mathcal{T})^{d\times d}_{\mathrm{sym}}) := \{a \in L^2(\Omega)^{d\times d} : \forall T \in \mathcal{T}, a|_T \in \operatorname{dev}\mathbb{R}^{d\times d}_{\mathrm{sym}}\}$$

Backward Euler scheme

The first time step problem

Find $X^1 = (U^1, (P^1_r)_{r \in I}) := (U^1, P^1) \in \mathcal{S}$ such that

$$\langle \ell(t_1), (Y - \frac{X^1 - X^0}{k_1}) \rangle \leq a(X^1, Y - \frac{X^1 - X^0}{k_1}) + \psi(Q) - \psi(\frac{P^1 - P^0}{k_1}).$$

holds for all $Y = (V, Q) = (V, (Q_r)_{r \in I}) \in \mathcal{S}$.

After introducing an incremental variable $X := (U, P) = X^1 - X^0$
and a linear functional $L(Y) = \langle \ell(t_1), Y \rangle - a(X^0, Y)$
we obtain a one-time step incremental problem

$$L(Y - X) \leq a(X, Y - X) + \psi(Q) - \psi(P) \quad \text{for all } Y = (V, Q) \in \mathcal{S}.$$

Introducing the energy functional

Lemma (Equivalent Reformulations)

For each $X = (U, P) \in \mathcal{S}$ the following three conditions (a)-(c) are equivalent:

$$(a) \quad L(Y - X) \leq a(X, Y - X) + \psi(Q) - \psi(P) \quad \forall Y = (V, Q) \in \mathcal{S}.$$

$$(b) \quad L(Y - X) = a(X, Y - X) \quad \text{for all } Y = (V, P) \in \mathcal{S} \quad \text{and}$$
$$L(Y - X) \leq a(X, Y - X) + \psi(Q) - \psi(P) \quad \forall Y = (U, Q) \in \mathcal{S}.$$

$$(c) \quad \Phi(X) = \min_{Y \in S} \Phi(Y) \quad \text{with } \Phi(Y) = \frac{1}{2} a(Y, Y) + \psi(Q) - L(Y).$$

Incremental plasticity: one time step with zero initial values

Problem (convex minimization problem)

$$\textit{Minimize} \quad \mathcal{H}(z) = \frac{1}{2}a(z; z) - b(z) + \psi(z) \quad \textit{over } z \in H$$

- $H = H^1_D \times L$ with L space of L^2 trace-free fields
- $a(w; z) = \langle \sigma, (\varepsilon(v) - q) \rangle + \langle \mathbb{H}p, q \rangle$ continuous and H-elliptic with $\sigma = \mathbb{C}(\varepsilon(u) - p), \quad w = (u, p), \quad z = (v, q)$
- $b \in H^*$
- $\psi(z) = \int_\Omega \sigma^y |q| \, d\Omega$ convex, l.s.c., positively homogeneous

Quasi Newton Solver for elastoplasticity

$\mathcal{H}(z)$ is convex but nonsmooth functional! How to minimize it?

New Contribution

- Found out, that $J(u) := \mathcal{H}(\varepsilon(u), \tilde{p}(\varepsilon(u)))$ is differentiable.
- Moreau 1965.
- Discretization of $\mathcal{D}J(u) = 0$ using FE-Method.
- Application of a Newton-Like Method ($\mathcal{D}^2 J$ does not exist).

Moreau regularization

Theorem (Moreau, 1965)

Let the function $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}}$ *be defined*

$$\mathcal{F}(x, y) = \frac{1}{2} \|y - x\|^2_{\mathcal{H}} + \psi(x) \tag{6}$$

where ψ *is a convex, proper and lower semi continuous mapping of* $\mathcal{H}$ *into* $\overline{\mathbb{R}}$. *Then*

$$F(y) := \inf_{x \in \mathcal{H}} \mathcal{F}(x, y)$$

is well defined as a functional from $\mathcal{H}$ *into* $\mathbb{R}$ *and there exists a unique mapping* $\tilde{x} : \mathcal{H} \to \mathcal{H}$ *such, that*

$$F(y) = \mathcal{F}(\tilde{x}(y), y)$$

holds for all $y \in \mathcal{H}$. *Moreover,* F *is strictly convex and Fréchet differentiable with the derivative*

$$\mathcal{D}F(y) = \langle y - \tilde{x}(y) \, , \, \cdot \rangle_{\mathcal{H}} \in \mathcal{H}^* \qquad \forall \, y \in \mathcal{H} \, . \tag{7}$$

Moreau regularization

Theorem of Moreau implies for elastoplasticity

Theorem

There is a unique function

$$p = p(\varepsilon(u))$$

and the energy functional

$$J(u) = \frac{1}{2}a(u, p(\varepsilon(u)); u, p(\varepsilon(u))) + \psi(p(\varepsilon(u))) - L(u)$$

is strictly convex and differentiable!

Slanted differentibiality

Definition (X. Chen, Z. Nashed, L. Qi — 2000)

A function $F : X \to Y$ is said to be *slantly differentiable at* x if

1. $\exists\ F^o : X \to L(X, Y)\ :$

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - F^o(x+h)\,h\|}{\|h\|} = 0\,.$$

2. F^o is uniformly bounded in an open neighbourhood of x.

$F^o(x)$ is said to be a *slanting function for* F *at* x.

Slanted differentibiality applied to elastoplasticity

Principles of the solution algorithms

- Use $(\mathcal{D}J)^o$ instead of $\mathcal{D}^2 J$ in Newton.
- Slant Newton Method (X. Chen, Z. Nashed, L. Qi — 2000).
- Showed superlinear convergence for the FE Problem.
- Showed superlinear convergence for the continuous problem, under special regularity assumptions on the Newton iterates.

One time step problem in Netgen/NgSolve: 3D

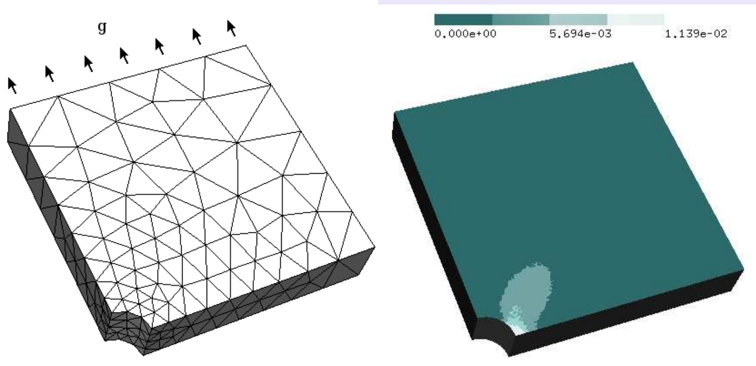


Figure: The coarsest FE-mesh (left) and the norm of the plastic strain field p (right) on a finer mesh.

Convergence Table – One Time Step

Level	0	...	3	4	5
DOF	245	...	14560	57920	231040
$\lvert u_j - u_{j-1} \rvert$:					
step 1	2.1826e-02	...	4.5238e-02	4.6300e-02	4.6603e-02
step 2	2.2225e-03	...	8.0839e-03	8.3886e-03	8.5454e-03
step 3	1.0478e-04	...	3.4440e-04	4.0032e-04	4.1602e-04
step 4	1.4404e-08	...	1.5206e-05	1.2050e-05	1.3944e-05
step 5	7.2634e-16	...	2.4947e-07	7.2972e-07	3.2631e-07
step 6		...	3.5062e-13	5.3972e-12	1.6473e-12
step 7		...		7.2441e-15	1.4518e-14

Table: The relative error in displacements.

1. superlinear convergence (or even quadratic) for each level
2. CG with a geometrical multigrid as a preconditioner to solve linear systems
3. number of nonlinear iterations almost mesh independent

One time step problem in Netgen/NgSolve: 3D

Figure: Crankshaft with elastic (blue) and two plastic phases (red and green). Computation with 808 448 tetrahedral elements took approx. 25 minutes.

Explaining papers to theory and numerics:

- Carsten Carstensen, Martin Brokate, Jan Valdman, A quasi-static boundary value problem in multi-surface elastoplasticity. I: Analysis. Math. Methods Appl. Sci. 27, No.14, 1697-1710 (2004)
- Carsten Carstensen, Martin Brokate, Jan Valdman, A quasi-static boundary value problem in multi-surface elastoplasticity. II: Numerical solution. Math. Methods Appl. Sci. 28, No.8, 881-901 (2005)
- Andreas Hofinger, Jan Valdman, Numerical solution of the two-yield elastoplastic minimization problem. Computing 81, No. 1, 35-52 (2007)
- Peter Gruber, Jan Valdman, Solution of one-time-step problems in elastoplasticity by a Slant Newton Method. SIAM J. Scientific Computing 31, No. 2, 1558-1580 (2009)

Elastoplasticity solver can be downloaded at

http://www.mathworks.com/matlabcentral/fileexchange/authors/37756

Outline

Stored energy and dissipation

Model by T. Roubicek (Prague):

STORED ENERGY at time t_1:

$$\mathcal{E}(U^1, \zeta^1, P_r^1) = \mathcal{E}_{elastic}(U^1, \zeta^1, P_r^1) + \mathcal{E}_{linearDamage}(\zeta^1) + \mathcal{E}_{gradientDamage}(\zeta^1)$$

and DISSIPATION POTENTIAL at time t_1:

$$\mathcal{R}(\zeta^1, \dot{U}^1, \dot{\zeta}^1, \dot{P}^1) = \mathcal{R}_{plastic}(\zeta^1, \dot{U}^1, \dot{\zeta}^1, \dot{P}^1) + \mathcal{F}(\dot{\zeta}^1) + \mathcal{R}_{gradientDamage}(\dot{\zeta}^1)$$

are approximated by the following two-step scheme:

PLASTICITY STEP: Given ζ^0, P_r^0, solve U^1, P_r^1 from the minimization problem

$$\mathcal{E}_{elastic}(U^1, \zeta^0, P_r^1) + \tau \mathcal{R}_{plastic}(\zeta^0, \frac{U^1 - U^0}{\tau}, 0, \frac{P_r^1 - P_r^1}{\tau}) \to \min,$$

where all terms are define as

1. elastic stored energy:

$$\mathcal{E}_{elastic}(U^1, \zeta^0, P_r^1) = \frac{1}{2} \int_\Omega \mathbb{C}(\zeta^0)(\varepsilon(U^1) - \sum_{r \in I} P_r^1) : (\varepsilon(U^1) - \sum_{r \in I} P_r^1) \, \mathrm{d}x$$

2. plastic dissipation potential

$$\tau \mathcal{R}_{plastic}(\zeta^0, \frac{U^1 - U^0}{\tau}, 0, \frac{P_r^1 - P_r^1}{\tau}) = \sum_{r \in I} \int_\Omega \sigma_r^y(\zeta^0) |P_r^1 - P_r^0| \, \mathrm{d}x \quad (8)$$

$$+ \sum_{r \in I} \frac{1}{2} \int_\Omega \frac{|P_r^1 - P_r^0|^2}{\tau} \, \mathrm{d}x \quad (9)$$

DAMAGE STEP: Given U^1, ζ^0, P_r^1, solve ζ^1 from the minimization problem

$$\mathcal{E}_{elastic}(U^1, \zeta^1, P_r^1) + \mathcal{E}_{linearDamage}(\zeta^1) + \mathcal{E}_{gradientDamage}(\zeta^1) + \tau\mathcal{F}\left(\frac{\zeta^1 - \zeta^0}{\tau}\right) \to \min \tag{10}$$

1. elastic stored energy:

$$\mathcal{E}_{elastic}(U^1, \zeta^1, P_r^1) = \frac{1}{2}\int_\Omega \mathbb{C}(\zeta^1)(\varepsilon(U^1) - \sum_{r\in I} P_r^1) : (\varepsilon(U^1) - \sum_{r\in I} P_r^1)\, \mathrm{d}x$$

2. linear damage stored energy: $\mathcal{E}_{linearDamage}(\zeta^1) = -c\int_\Omega \zeta^1\, \mathrm{d}x$
3. gradient damage stored energy: $\mathcal{E}_{gradientDamage}(\zeta^1) = \frac{\kappa_0}{2}\int_\Omega |\nabla\zeta^1|^2\, \mathrm{d}x$
4. nonlinear damage dissipation potential

$$\tau\mathcal{F}\left(\frac{\zeta^1 - \zeta^0}{\tau}\right) = \int_\Omega \left(\frac{a}{2}\frac{(|\zeta^1 - \zeta^0)^+|^2}{\tau} + \frac{b}{2}\frac{(|\zeta^1 - \zeta^0)^-|^2}{\tau} - d(\zeta^1 - \zeta^0)^-\right)\, \mathrm{d}x$$

Thank you for your attention!